

Aerodynamic Lift and Moment Calculations Using a Closed-Form Solution of the Possio Equation

*Jensen Lin
Flight Systems Research Center
University of California, Los Angeles
Los Angeles, California*

*Kenneth W. Iliff
NASA Dryden Flight Research Center
Edwards, California*

National Aeronautics and
Space Administration

Dryden Flight Research Center
Edwards, California 93523-0273

NOTICE

Use of trade names or names of manufacturers in this document does not constitute an official endorsement of such products or manufacturers, either expressed or implied, by the National Aeronautics and Space Administration.

Available from the following:

NASA Center for AeroSpace Information (CASI)
7121 Standard Drive
Hanover, MD 21076-1320
(301) 621-0390

National Technical Information Service (NTIS)
5285 Port Royal Road
Springfield, VA 22161-2171
(703) 487-4650

ABSTRACT

In this paper, we present closed-form formulas for the lift and moment coefficients of a lifting surface in two-dimensional, unsteady, compressible, subsonic flow utilizing a newly developed explicit analytical solution of the Possio equation. Numerical calculations are consistent with previous numerical tables based on series expansions or ad hoc numerical schemes. More importantly, these formulas lend themselves readily to flutter analysis, compared with the tedious table-look-up schemes currently in use.

NOMENCLATURE

a	location of the elastic axis
a_{∞}	speed of sound, ft/sec
A	doublet intensity
b	half chord, ft
C	Theodorsen function
Ch	linear functional
d	derivative with respect to η
D	determinant of a linear algebraic equation
e	exponential constant
E	linear functional
EI	elastic rigidity, lb ft
f	function
$f(M)$	function of Mach
G	kernel of Possio integral equation
GJ	torsional rigidity, lb ft
h	plunging motion, ft
i	$\sqrt{-1}$
Im	the imaginary part of a complex number
I_y	mass moment of inertia about the elastic axis, slug/ft
k	reduced frequency = $\frac{\omega b}{U}$
\bar{k}	$\frac{\lambda b}{U}$, reduced frequency
K	function
K_n	modified Bessel functions of the second kind and of order n
l	span of the wing, ft
\lim	limit

L	lift
L_h	coefficient of lift due to plunge motion
L_α	coefficient of lift due to pitch motion
m	mass per unit span, slug/ft
M	Mach number = $\frac{U}{a_\infty}$
M	moment
M_h	coefficient of moment due to plunge motion
M_α	coefficient of moment due to pitch motion
p	function
P-G transformation	Prandtl-Glauert transformation
q	function
r	function
Re	the real part of a complex number
Sh	linear functional
S_y	static mass moment per unit length about the elastic axis, slug/ft/ft
t	time
U	free-stream velocity, ft/sec
w_a	downwash, ft/sec
x, y, z	Cartesian coordinates
α	pitching motion, angle of attack
β	$\sqrt{1 - M^2}$
γ	$M \frac{\lambda b}{U\beta^2} = M\mu = \frac{M}{\beta^2} \bar{k}$, scaled reduced frequency
ΔP	pressure distribution = ρUA
ζ	variable
η	dummy integration variable
λ	frequency, rad
μ	$\frac{\lambda b}{U\beta^2} = \frac{\bar{k}}{\beta^2}$, compressible reduced frequency
ρ	air density, slug/ft ³

σ	dummy variable
ϕ	velocity potential
χ	dummy variable
ω	frequency of oscillation, rad/sec

1. INTRODUCTION

The pressure distribution on a lifting surface in unsteady aerodynamics and the lift/moment coefficients deduced therefrom are essential for many aerodynamic/aeroelastic calculations including flutter analysis. A central role is played in this theory by the Possio (integral) equation, relating the normal velocity of the fluid (downwash) to the pressure distribution of a lifting surface in two-dimensional, oscillatory, subsonic compressible flow, derived by Possio.¹ Despite the effort of many aerodynamicists and aeroelasticians, however, no explicit solution has so far been found. Dietze² and later Fettis³ and others turned to creating tables of lift and moment coefficients based on various numerical approximations. The lack of explicit formulas for the aerodynamic coefficients, in turn, resulted in making the aeroelastic calculations a tedious iterative process. Haskind,⁴ Reissner,⁵ and Timman, et al.,⁶ formulated the two-dimensional flow problem in elliptic coordinates and this resulted in a series expansion in terms of Mathieu functions and also developed aerodynamic coefficient tables. By the 1950's, the aerodynamic theory had extended to the more general three-dimensional lifting surface problem. Watkins, Runyan and Woolston⁷ used series expansions which led to a numerical scheme known as the Kernel Method (KFM). In the late 1960's, Albano and Rodden⁸ developed a numerical scheme called the "doublet-lattice method" which extends to even more complicated nonplanar configurations, but without theoretical justification.^{8,9} As the flow problem became more complicated it became even more difficult to combine the structure with the aerodynamics necessary for the aeroelastic problem.

Recently, Balakrishnan¹⁰ derived an explicit solution to the Possio equation, which is accurate to the order of $\frac{M^2}{1-M^2} \log \frac{M}{1-M^2}$. This approximation is certainly valid for $M < 0.7$, above which the linearized theory is generally conceded as no longer valid. Figure 1 shows that $\frac{M^2}{1-M^2} \log \frac{M}{1-M^2}$ is well below one for $M < 0.7$. In this paper we derive explicit formulas for the lift and moment for any two-dimensional lifting surface in subsonic compressible flow, based on the Balakrishnan solution.

We begin in section 2 with the Laplace transform version of the Possio equation and its solution derived by Balakrishnan.¹⁰ In section 3, we proceed to obtain closed-form formulas for the lift and moment. Numerical calculations are given in section 4, which are compared with results obtained by previous authors,^{2,3,6,11} showing acceptable agreement. In section 5 we illustrate the ease with which the lift and moment formulas are applied to flutter analysis as compared with table-look-up schemes, by considering a standard model for wings with large aspect ratio. Concluding remarks are in the final section.

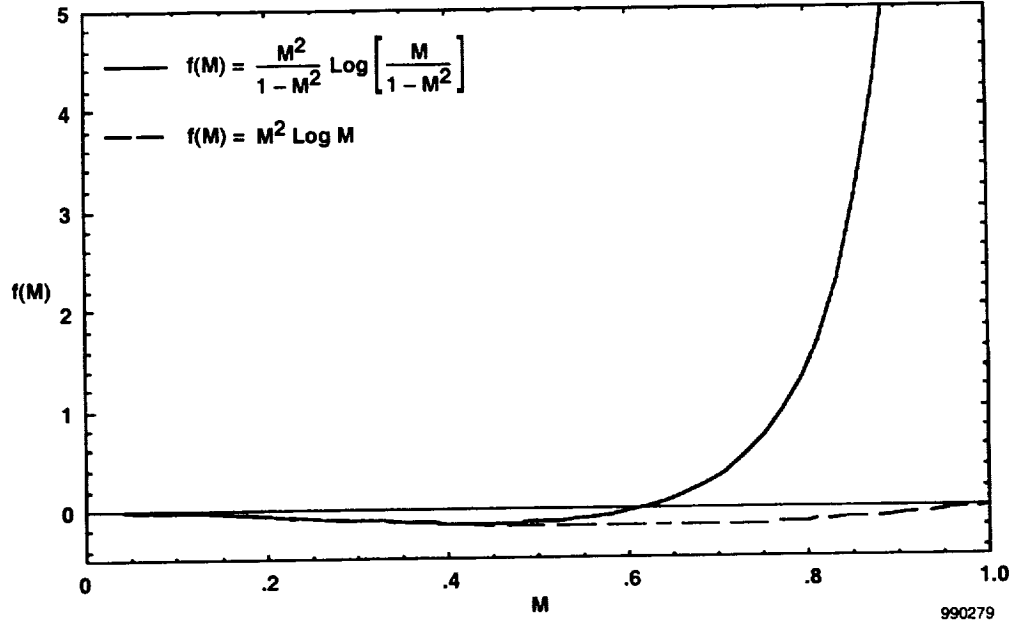


Figure 1. Plot of $\frac{M^2}{1-M^2} \log \frac{M}{1-M^2}$ and $M^2 \log M$.

2. SOLUTION OF POSSIO EQUATION – LAPLACE TRANSFORM VERSION

In this section we present the Laplace transform version of the Possio equation and its solution, referring to reference 10 for more detailed analysis. To clarify the terminology, we begin with the two-dimensional field equations for the velocity potential in (linearized) compressive flow:

$$\frac{1}{a_\infty^2} \frac{\partial^2 \phi(x, z, t)}{\partial t^2} + 2U \frac{\partial^2 \phi(x, z, t)}{\partial t \partial x} - \beta^2 \frac{\partial^2 \phi(x, z, t)}{\partial x^2} - \frac{\partial^2 \phi(x, z, t)}{\partial z^2} = 0 \quad (2.1)$$

$$-\infty < x < \infty, 0 \leq z < \infty,$$

subject to the boundary conditions:

i) *Flow Tangency*

$$\frac{\partial \phi(x, 0^+, t)}{\partial z} = w_a(x, t), \quad |x| < b; \quad (2.2)$$

ii) *Kutta-Joukowski*

$$\frac{\partial \phi(x, 0^+, t)}{\partial t} + U \frac{\partial \phi(x, 0^+, t)}{\partial x} = 0, \quad x = b^- \text{ and } x \geq b; \quad (2.3)$$

iii) and the usual vanishing far-field conditions.

A “particular” solution of (2.1), which satisfies *iii*) and *ii*) for $x \geq b$ is given as (see ref. 12)

$$\phi(x, z, t) = -\frac{1}{4\pi} \int_{-b}^b d\zeta \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{x-\zeta} \frac{\partial}{\partial z} \left[\frac{A\left(\zeta, t + \frac{\chi}{U\beta^2} - \frac{x-\zeta}{U} - \frac{\sqrt{\chi^2 + \beta^2(\eta^2 + z^2)}}{a_\infty \beta^2}\right)}{\sqrt{\chi^2 + \beta^2(\eta^2 + z^2)}} \right] d\chi. \quad (2.4)$$

It is a “particular” solution in that the initial conditions are assumed to be:

$$\phi(x, z, 0) = \frac{\partial \phi(x, z, 0)}{\partial t} = 0,$$

which are necessary for the Laplace transform theory. To satisfy the *flow-tangency* condition, we substitute this solution into (2.2), and must obtain

$$w_a(x, t) = \lim_{z \rightarrow 0} \frac{-1}{4\pi} \int_{-b}^b d\zeta \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{x-\zeta} \frac{\partial^2}{\partial z^2} \left[\frac{A\left(\zeta, t + \frac{\chi}{U\beta^2} - \frac{x-\zeta}{U} - \frac{\sqrt{\chi^2 + \beta^2(\eta^2 + z^2)}}{a_\infty \beta^2}\right)}{\sqrt{\chi^2 + \beta^2(\eta^2 + z^2)}} \right] d\chi. \quad (2.5)$$

Next, we take the Laplace transform on both sides by defining

$$\hat{w}_a(x, \lambda) = \int_0^\infty e^{-\lambda t} w_a(x, t) dt; \quad \hat{A}(x, \lambda) = \int_0^\infty e^{-\lambda t} A(x, t) dt, \quad \text{Re } \lambda > 0.$$

We now have:

$$\begin{aligned} \hat{w}_a(x, \lambda) &= \lim_{z \rightarrow 0} \frac{-1}{4\pi} \int_{-b}^b d\zeta \int_{-\infty}^{x-\zeta} d\chi \cdot \exp\left(\lambda \left(\frac{\chi}{U\beta^2} - \frac{x-\zeta}{U}\right)\right) \cdot \hat{A}(\zeta, \lambda) \\ &\quad \cdot \frac{\partial^2}{\partial z^2} \int_{-\infty}^{\infty} \frac{\exp\left(-\lambda \frac{\sqrt{\chi^2 + \beta^2(\eta^2 + z^2)}}{a_\infty \beta^2}\right)}{\sqrt{\chi^2 + \beta^2(\eta^2 + z^2)}} d\eta \\ &= \frac{-1}{4\pi} \int_{-b}^b d\zeta \int_{-\infty}^{x-\zeta} d\chi \cdot \exp\left(\lambda \left(\frac{\chi}{U\beta^2} - \frac{x-\zeta}{U}\right)\right) \cdot \hat{A}(\zeta, \lambda) \cdot \lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} K_0\left(\lambda \frac{\sqrt{\chi^2 + \beta^2 z^2}}{a_\infty \beta^2}\right). \end{aligned} \quad (2.6)$$

This leads to (see ref. 10) the “Laplace transform version” of the Possio equation,

$$\hat{w}_a(x, \lambda) = \frac{\lambda}{\rho U^2} \int_{-b}^b (G(x - \zeta, M, \lambda) \Delta \hat{P}(\zeta, \lambda) d\zeta, \quad (2.7)$$

where

$$\begin{aligned}\Delta\hat{P}(\zeta, \lambda) &= -\rho U \hat{A}(\zeta, \lambda), \\ G(y, M, \lambda) &= \frac{1}{2\pi\beta} \left\{ \exp\left(\frac{\lambda M^2}{U\beta^2} y\right) \left[\frac{M|y|}{y} K_1\left(\frac{\lambda M}{U\beta^2} |y|\right) + K_0\left(\frac{\lambda M}{U\beta^2} |y|\right) \right] \right. \\ &\quad \left. - \frac{\lambda}{U} \int_{-\infty}^y \exp\left(\frac{-\lambda y}{U} + \frac{\lambda \sigma}{U\beta^2}\right) K_0\left(\frac{\lambda M|\sigma|}{U\beta^2}\right) d\sigma \right\}.\end{aligned}$$

$G(y, M, \lambda)$ is analytic in λ for the entire complex plane except for the “branch cut” along the negative real axis, due to the essential singularity of the modified Bessel functions. Hence, we can set $\lambda = i\omega$, yielding the usual Possio equation for “oscillatory motion.”^{1, 13}

The explicit solution to (2.7) developed by Balakrishnan in ref. 10 is given by

$$f(x, \lambda) = r(x, \lambda) + \mu \int_{-1}^x \cosh \gamma(x - \sigma) r(\sigma, \lambda) d\sigma + \gamma \int_{-1}^x \sinh \gamma(x - \sigma) r(\sigma, \lambda) d\sigma \quad (2.8)$$

where

$$\begin{aligned}r(x, \lambda) &= g(x, \lambda) - \gamma K(\gamma, x) \int_{-1}^1 \sinh \gamma(1 - \sigma) r(\sigma, \lambda) d\sigma \\ &\quad + (\bar{k} p(\mu, x) - M \gamma K(\gamma, x)) \int_{-1}^1 \cosh \gamma(1 - \sigma) r(\sigma, \lambda) d\sigma,\end{aligned} \quad (2.8a)$$

and

$$\begin{aligned}f(x, \lambda) &= e^{-M\gamma x} \Delta\hat{P}(x, \lambda), \\ g(x, \lambda) &= \rho U \frac{2}{\pi\beta} \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \sqrt{\frac{1+\zeta}{1-\zeta}} \frac{e^{-M\gamma\zeta} \hat{w}_a(\zeta, \lambda)}{\zeta - x} d\zeta, \\ K(\gamma, x) &= \frac{1}{\pi^2} \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \sqrt{\frac{1+\zeta}{1-\zeta}} \frac{K_0(\gamma(1-\zeta))}{\zeta - x} d\zeta, \\ p(\mu, x) &= \frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} \int_0^\infty e^{-\mu t} \frac{1}{x-t-1} \sqrt{\frac{2+t}{t}} dt.\end{aligned}$$

For ease of notation, we have set $b = 1$ since it is only a matter of scaling. The coefficients

$$\int_{-1}^1 \sinh \gamma(1 - \sigma) r(\sigma) d\sigma \text{ and } \int_{-1}^1 \cosh \gamma(1 - \sigma) r(\sigma) d\sigma$$

are readily calculated. Let

$$Sh(q) = \int_{-1}^1 \gamma \sinh \gamma (1 - \sigma) q(\sigma) d\sigma$$

$$Ch(q) = \int_{-1}^1 \cosh \gamma (1 - \sigma) q(\sigma) d\sigma.$$

Then from (2.8a) we obtain the linear equations:

$$\begin{aligned} Sh(r) &= Sh(g) - Sh(K)Sh(r) + [\bar{k}Sh(p) - M\gamma Sh(K)]Ch(r) \\ Ch(r) &= Ch(g) - Ch(K)Sh(r) + [\bar{k}Ch(p) - M\gamma Ch(K)]Ch(r). \end{aligned} \quad (2.9)$$

The determinant is

$$D = (1 + Sh(K))(1 - \bar{k}Ch(p)) + (M\gamma + \bar{k}Sh(p))Ch(K).$$

A plot of D as a function of the reduced frequency is given in figure 2 for a family of values of $M \leq 0.7$ showing it is actually positive for these values, so that (2.9) has a unique solution.

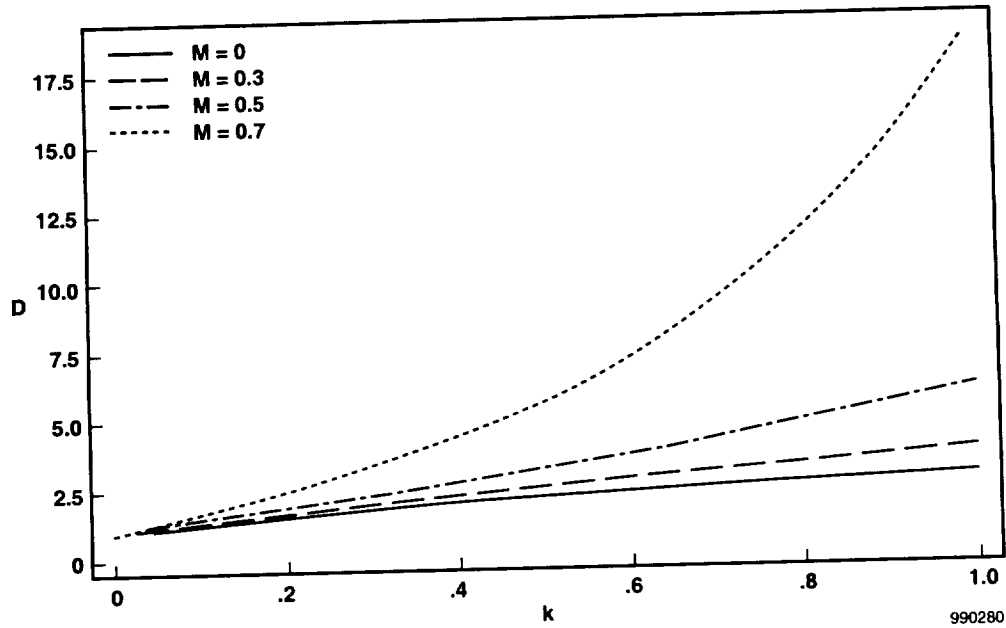


Figure 2. The determinant D as a function of the reduced frequency (k) for various Mach numbers.

For the incompressible case, the Balakrishnan solution simplifies this and we obtain

$$f(x, \lambda) = r(x, \lambda) + \bar{k} \int_{-1}^x r(\sigma, \lambda) d\sigma \quad (2.10)$$

and

$$r(x) = g(x) + \bar{k}p(\bar{k}, x) \int_{-1}^1 r(\sigma) d\sigma, \quad (2.11)$$

where

$$\int_{-1}^1 r(\sigma) d\sigma = \frac{\int_{-1}^1 g(\sigma) d\sigma}{1 - \bar{k} \int_{-1}^1 p(\bar{k}, \sigma) d\sigma} = \frac{1}{\bar{k} K_0(\bar{k}) + K_1(\bar{k})} \int_{-1}^1 g(\sigma) d\sigma = \frac{1}{D(\bar{k})} \int_{-1}^1 g(\sigma) d\sigma.$$

Plugging (2.11) into (2.10), we have

$$f(x, \lambda) = g(x) + \bar{k} \int_{-1}^x g(\sigma) d\sigma + \bar{k} \frac{1}{D(\bar{k})} \int_{-1}^x g(\sigma) d\sigma \left[p(\bar{k}, x) + \bar{k} \int_{-1}^x p(\bar{k}, \sigma) d\sigma \right], \quad (2.12)$$

which agrees with the known (e.g. Küssner-Schwarz¹⁴) solution for “oscillatory motion.”

3. LIFT AND MOMENT CALCULATIONS

In this section we calculate the lift and moment based on Balakrishnan’s solution, equation (2.8), accurate again to the order of $\frac{M^2}{1-M^2} \log \frac{M}{1-M^2}$ (or equivalently $M^2 \log M$).

The lift is given by:

$$\begin{aligned} \hat{L}(\lambda) &= -\int_{-1}^1 \Delta \hat{P}(x, \lambda) dx \\ &= -\int_{-1}^1 e^{M\gamma x} \left[r(x) + \mu \int_{-1}^x \cosh \gamma(x - \sigma) r(\sigma) d\sigma + \gamma \int_{-1}^x \sinh \gamma(x - \sigma) r(\sigma) d\sigma \right] dx \\ &= -E_0(r) - \frac{e^{M\gamma}}{M\gamma} Sh(r), \end{aligned} \quad (3.1)$$

where

$$E_0(r) = \int_{-1}^1 e^{M\gamma x} r(x) dx.$$

Similarly we have for the moment about the axis $x = a$ as,

$$\begin{aligned}
M(\lambda) &= \int_{-1}^1 (x-a) \Delta \hat{P}(x, \lambda) dx \\
&= \int_{-1}^1 e^{M\gamma x} (x-a) \left[r(x) + \mu \int_{-1}^x \cosh \gamma(x-\sigma) r(\sigma) d\sigma \right. \\
&\quad \left. + \gamma \int_{-1}^x \sinh \gamma(x-\sigma) r(\sigma) d\sigma \right] dx \\
&= \left(\frac{1}{M^2 \bar{k}} - a \right) E_0(r) + E_1(r) - \frac{1}{M^2 \bar{k}} e^{M\gamma} Ch(r) \\
&\quad + \frac{1}{M\gamma} \left(1 + \frac{1}{\bar{k}} - a \right) e^{M\gamma} Sh(r),
\end{aligned} \tag{3.2}$$

where

$$E_1(r) = \int_{-1}^1 x e^{M\gamma x} r(x) dx.$$

Hence to calculate lift and moment we need to calculate two more constants namely,

$$E_0(r) \text{ and } E_1(r).$$

We have again from (2.8a),

$$E_0(r) = E_0(g) - E_0(K) Sh(r) + [\bar{k} E_0(p) - M\gamma E_0(K)] Ch(r),$$

$$E_1(r) = E_1(g) - E_1(K) Sh(r) + [\bar{k} E_1(p) - M\gamma E_1(K)] Ch(r).$$

We are then left to find the twelve constants, $E_0(g)$, $E_1(g)$, $Sh(g)$, $Ch(g)$, $E_0(p)$, $E_1(p)$, $Sh(p)$, $Ch(p)$, $E_0(K)$, $E_1(K)$, $Sh(K)$ and $Ch(K)$. It is difficult to obtain explicit formulas for these constants. But since we are only interested in those terms of the order $M^2 \log M$, we expand them in a power series of M . Keeping terms up to $M^2 \log M$, we obtain

$$\hat{L}(\lambda) = \hat{L}_0(\lambda) + (M^2 \log M) \hat{L}_M(\lambda) \text{ and } \hat{M}(\lambda) = \hat{M}_0(\lambda) + (M^2 \log M) \hat{M}_M(\lambda),$$

where

$$\begin{aligned}
\hat{L}_0(\lambda) &= -2\rho U \int_{-1}^1 \sqrt{\frac{1+\sigma}{1-\sigma}} [\bar{k}(1-\sigma) + C(\bar{k})] \hat{w}_a(\sigma, \lambda) d\sigma, \\
\hat{M}_0(\lambda) &= \rho U \int_{-1}^1 \left(\sqrt{\frac{1+\sigma}{1-\sigma}} \{-1 - 2a\bar{k} - (1+2a)C(\bar{k}) + [2 + \bar{k}(1+2a)]\sigma - \bar{k}\sigma^2\} \hat{w}_a(\sigma, \lambda) \right) d\sigma, \\
\hat{L}_M(\lambda) &= -\rho U \int_{-1}^1 \sqrt{\frac{1+\sigma}{1-\sigma}} \{\bar{k}^3 + 3\bar{k}^2 C(\bar{k}) + 2\bar{k}C(\bar{k})^2 - \sigma[\bar{k}^3 + 2\bar{k}^2 C(\bar{k})]\} \hat{w}_a(\sigma, \lambda) d\sigma, \\
\hat{M}_M(\lambda) &= \rho U \int_{-1}^1 \sqrt{\frac{1+\sigma}{1-\sigma}} \{-a\bar{k}^3 - (1+3a)\bar{k}^2 C(\bar{k}) - (1+2a)\bar{k}C(\bar{k})^2 \\
&\quad + \sigma[a\bar{k}^3 + (1+2a)\bar{k}^2 C(\bar{k})]\} \hat{w}_a(\sigma, \lambda) d\sigma
\end{aligned} \tag{3.3}$$

and

$$C(\bar{k}) = \frac{K_1(\bar{k})}{K_1(\bar{k}) + K_0(\bar{k})} = \text{Theodorsen function.}$$

Equation (3.3) is for a general class of lifting surfaces, in that the downwash function is arbitrary. For a typical section we can express the downwash as

$$w_a(x, t) = -h(t) - (x-a)\dot{\alpha}(t) - U\alpha(t), \text{ for } |x| \leq b,$$

or

$$\hat{w}_a(\lambda) = -\hat{h}(\lambda) - (x-a)\lambda\hat{\alpha}(\lambda) - U\hat{\alpha}(\lambda). \tag{3.4}$$

We ignore the initial conditions here since we will be interested only in the stability of the wing. We could also easily include control surfaces, but for simplicity we will not do so.

Plugging (3.4) into (3.3) we obtain:

$$\begin{aligned}
\hat{L}(\lambda) &= \pi\rho U^2 b \left[L_h(\bar{k}) \frac{\hat{h}(\lambda)}{b} + L_\alpha(\bar{k}) \hat{\alpha}(\lambda) \right], \\
\hat{M}(\lambda) &= \pi\rho U^2 b^2 \left[M_h(\bar{k}) \frac{\hat{h}(\lambda)}{b} + M_\alpha(\bar{k}) \hat{\alpha}(\lambda) \right],
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
L_h(\bar{k}) &= \bar{k}^2 + 2\bar{k}C(\bar{k}) + M^2 \log M \left[\frac{\bar{k}^4}{2} + 2\bar{k}^3 C(\bar{k}) + 2\bar{k}^2 C(\bar{k})^2 \right], \\
L_\alpha(\bar{k}) &= \bar{k} - a\bar{k}^2 + C(\bar{k})[2 + (1 - 2a)\bar{k}] \\
&\quad + M^2 \log M \left\{ \frac{\bar{k}^3}{2} - \frac{\bar{k}^4}{2}a + C(\bar{k}) \left[2\bar{k}^2 + \left(\frac{1}{2} - 2a \right) \bar{k}^3 \right] + C(\bar{k})^2 [2\bar{k} + (1 - 2a)\bar{k}^2] \right\}, \\
M_h(\bar{k}) &= a\bar{k}^2 + (1 + 2a)\bar{k}C(\bar{k}) + M^2 \log M \left\{ (1 + 2a)\bar{k}^2 C(\bar{k})^2 + \left(\frac{1}{2} + 2a \right) \bar{k}^3 C(\bar{k}) + \frac{\bar{k}^4}{2}a \right\}, \\
M_\alpha(\bar{k}) &= \left(a - \frac{1}{2} \right) \bar{k} - \left(\frac{1}{8} + a^2 \right) \bar{k}^2 + C(\bar{k}) \left[1 + 2a + \left(\frac{1}{2} - 2a^2 \right) \bar{k} \right] + M^2 \log M \left\{ \frac{a}{2} \bar{k}^3 - \frac{a^2}{2} \bar{k}^4 \right. \\
&\quad \left. + C(\bar{k})^2 \left[(1 + 2a)\bar{k} + \left(\frac{1}{2} - 2a^2 \right) \bar{k}^2 \right] + C(\bar{k}) \left[\left(\frac{1}{2} + 2a \right) \bar{k}^2 - 2a^2 \bar{k}^3 \right] \right\}.
\end{aligned} \tag{3.6}$$

Smilg and Wassermann¹⁵ defined the aerodynamic coefficients differently, they had

$$\begin{aligned}
\hat{L}(\lambda) &= -\pi\rho U^2 b \bar{k}^2 \left\{ \tilde{L}_h(\bar{k}) \frac{\hat{h}(\lambda)}{b} + \left[\tilde{L}_\alpha(\bar{k}) - \left(\frac{1}{2} + a \right) \tilde{L}_h(\bar{k}) \right] \hat{\alpha}(\lambda) \right\}, \\
\hat{M}(\lambda) &= -\pi\rho U^2 b^2 \bar{k}^2 \left\{ \left[\tilde{M}_h(\bar{k}) - \left(\frac{1}{2} + a \right) \tilde{L}_h(\bar{k}) \right] \frac{\hat{h}(\lambda)}{b} \right. \\
&\quad \left. + \left[\tilde{M}_\alpha \bar{k} - \left(\frac{1}{2} + a \right) (\tilde{L}_\alpha(\bar{k}) + \tilde{M}_h(\bar{k})) + \left(\frac{1}{2} + a \right)^2 \tilde{L}_h(\bar{k}) \right] \hat{\alpha}(\lambda) \right\}.
\end{aligned}$$

Here the coefficients are given by:

$$\begin{aligned}
\tilde{L}_h(\bar{k}) &= 1 + \frac{2}{\bar{k}}C(\bar{k}) + M^2 \log M \left[\frac{\bar{k}^2}{2} + 2\bar{k}C(\bar{k}) + 2C(\bar{k})^2 \right], \\
\tilde{L}_\alpha(\bar{k}) &= \frac{1}{2} + \frac{1}{\bar{k}} + C(\bar{k}) \left[\frac{2}{\bar{k}^2} + \frac{2}{\bar{k}} \right] + M^2 \log M \left\{ \frac{\bar{k}}{2} + \frac{\bar{k}^2}{4} + C(\bar{k}) \left[2 + \frac{3}{2}\bar{k} \right] + C(\bar{k})^2 \left[2 + \frac{2}{\bar{k}} \right] \right\}, \\
\tilde{M}_h(\bar{k}) &= \frac{1}{2} + M^2 \log M \left\{ \frac{\bar{k}^2}{4} + \frac{\bar{k}}{2}C(\bar{k}) \right\}, \\
\tilde{M}_\alpha(\bar{k}) &= \frac{3}{8} + \frac{1}{\bar{k}} + M^2 \log M \left\{ \frac{\bar{k}}{4} + \frac{\bar{k}^2}{8} + C(\bar{k}) \left[\frac{1}{2} + \frac{\bar{k}}{2} \right] \right\}.
\end{aligned} \tag{3.7}$$

For $a = -\frac{1}{2}$, we have

$$L_h(\bar{k}) = \bar{k}^2 \tilde{L}_h(\bar{k}), \quad L_\alpha(\bar{k}) = \bar{k}^2 \tilde{L}_\alpha(\bar{k}), \quad M_h(\bar{k}) = -\bar{k}^2 \tilde{M}_h(\bar{k}), \quad M_\alpha(\bar{k}) = -\bar{k}^2 \tilde{M}_\alpha(\bar{k}).$$

Note: In the asymptotic expansions of the modified Bessel functions, we suppressed the dependence of γ on the reduced frequency (\bar{k}). The dependence is such that the higher the value of \bar{k} , the smaller the value of M will need to be. However, for most aeroelastic calculations, we are only interested in the first few modes where $|\bar{k}|$ is < 1 . In fact we have

$$\gamma = M \frac{\lambda b}{U \beta^2} = \frac{1}{\beta^2} \frac{\lambda b}{a_\infty} = \frac{1}{\beta^2} \bar{k}_\infty \approx \bar{k}_\infty \left(1 + \frac{M^2}{2} \right) \leq \bar{k} \left(1 + \frac{M^2}{2} \right). \tag{3.8}$$

4. COMPARISON WITH PREVIOUS RESULTS

The definitions of the lift and moment coefficients in (3.6) are similar to those adopted by Possio, Dietze, and Timman. Timman's lift and moment coefficients are defined for rotation about the midchord, $a = 0$, hence his L_α . These moment coefficients are different from equation (3.6) and Dietze's values. For $M = 0.5$, figures 3–6 compare the aerodynamic coefficients in equation (3.6) with their corresponding values found in Dietze and Timman. We also included the Prandtl-Glauert transformation of (3.6), i.e. all coefficients are divided by β . As $k \rightarrow 0$, the Prandtl-Glauert version will reach the correct steady state value. Upon close inspection we found that Dietze's L_h and L_α were missing $-k^2$ and $-k^2/2$ respectively. When these errors are taken into account, our results show good agreement with Dietze's values. The differences between Dietze's values and ours are all within 10 percent. With the adjustment of the rotation axis, the differences between Timman's values and ours are also within 10 percent, except at high reduced frequencies. At these high reduced frequencies Fettis found that Timman's values are erroneous.¹⁶ As for the moment coefficients, again Dietze left out some terms there as well. When these errors are taken into account the difference between Dietze's numbers and ours are again within 10 percent (not shown).

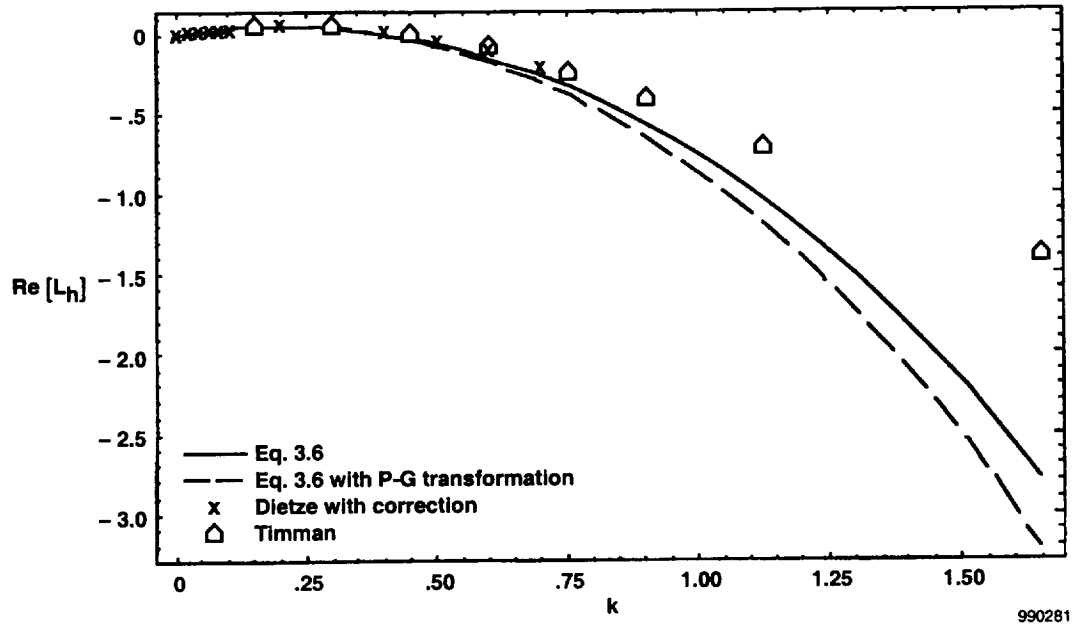


Figure 3. Lift coefficient due to plunge motion as a function of reduced frequency (k) with $M = 0.5$ (real part).

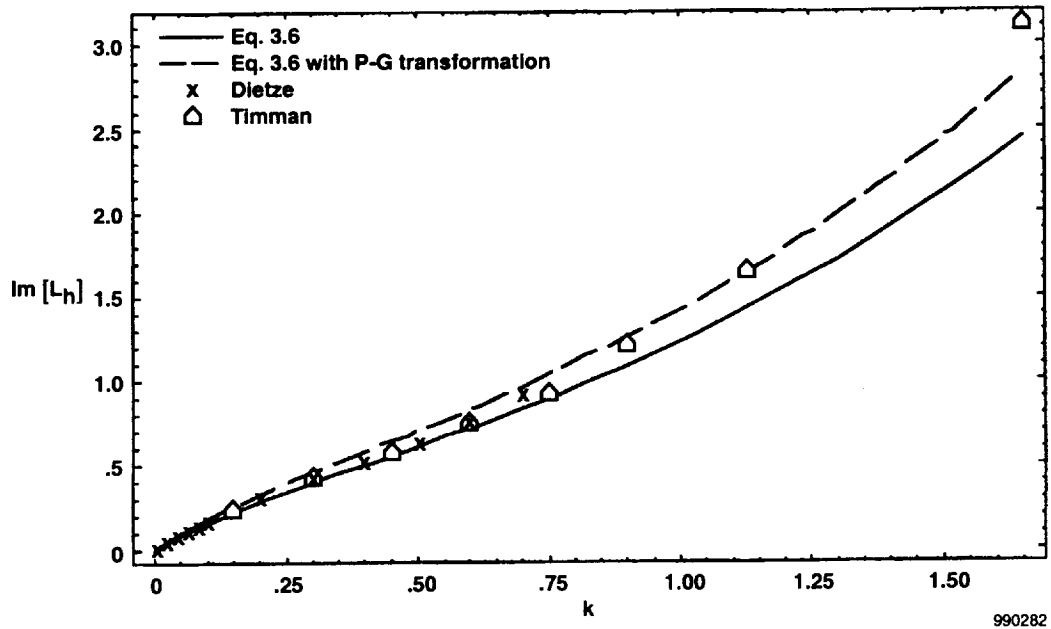


Figure 4. Lift coefficient due to plunge motion as a function of reduced frequency (k) with $M = 0.5$ (imaginary part).

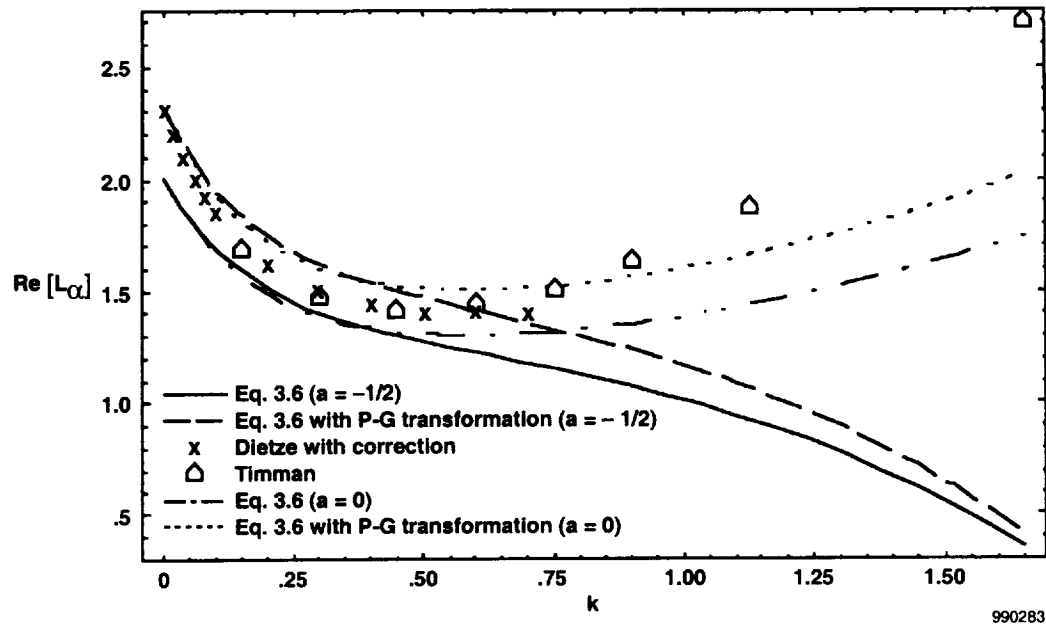


Figure 5. Lift coefficient due to pitch motion as a function of reduced frequency (k) with $M = 0.5$ (real part).

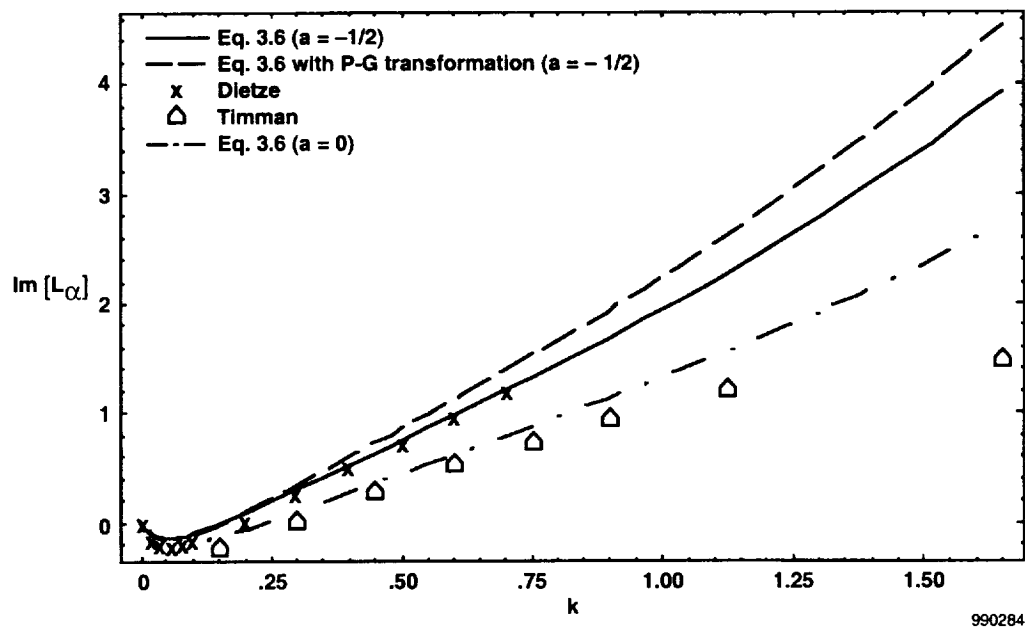


Figure 6. Lift coefficient due to pitch motion as a function of reduced frequency (k) with $M = 0.5$ (imaginary part).

For the case where $M = 0.7$, figures 7–10 compare results from equation (3.7) with those that belong to Fettis² and Blanch¹¹ whenever they follow Smilg and Wassermann notations. Blanch employed Reissner's method to calculate the aerodynamic coefficients, whereas Fettis approximated Possio's equation in obtaining his results. It is uncanny that the values are within a few hundredths of each other. Our numbers compare reasonably well with theirs for $\text{Im}[L_h]$ and $\text{Re}[L_\alpha]$ shown in figures 8 and 9 respectively. The differences are close to 15 to 20 percent. However, for $\text{Re}[L_h]$ and $\text{Im}[L_\alpha]$, there is nearly a 60-percent difference between our results and theirs. For this particular Mach number Fettis' values agreed with Dietze's when he took into account Dietze's missing factors mentioned earlier. There were some difficulties in the convergence of Dietze's iteration scheme for such a high Mach number. Fettis bypassed these difficulties by approximating Possio's integral equation with a polynomial. He expanded the kernel function in a power series of k after he had removed the incompressible part. On the other hand, Blanch followed Reissner's method of expansions in terms of the Mathieu functions. Hence, their approximations are no longer valid for large Mach numbers and large reduced frequencies. Neither Fettis nor Blanch indicated how many terms should be taken in their expansions or what the limiting values are for the Mach number or the reduced frequency in their approximations. Our results are also approximations, but they are analytic rather than numerical approximations. If our approximations are good enough for $\text{Im}[L_h]$ and $\text{Re}[L_\alpha]$, they should be good enough for $\text{Re}[L_h]$ and $\text{Im}[L_\alpha]$ as well. The differences are even greater in the moment coefficients. These differences are 60-percent and beyond (not shown). Just as with lift coefficients, we simply cannot conclude which approximation is better. From (3.7) it follows that

$$\tilde{M}_h(\bar{k}) + \tilde{L}_\alpha(\bar{k}) = \left(1 + \frac{1}{\bar{k}}\right) \tilde{L}_h(\bar{k}),$$

which is a formula derived by Fettis in reference 15. This relation is exact and was derived analytically based solely on the kinematics of the airfoil and the existence and uniqueness of the solution to the Possio equation. Overall, for $M < 0.7$ and $k < 1$, our approximations are in good agreement with previous tabulated values.

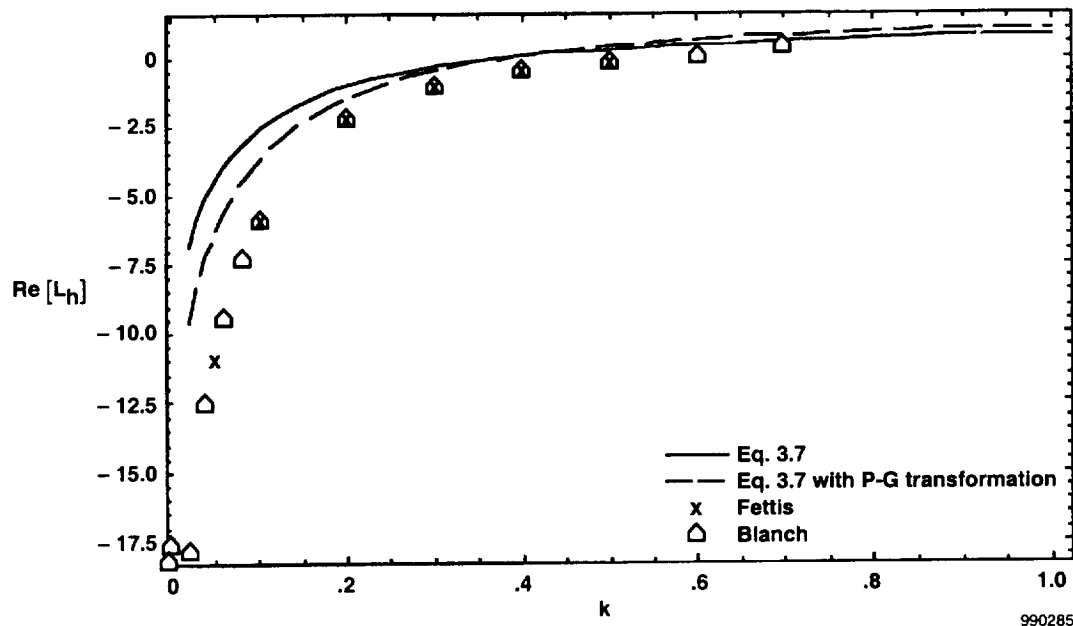


Figure 7. Lift coefficient due to plunge motion as a function of reduced frequency (k) with $M = 0.7$ (real part).

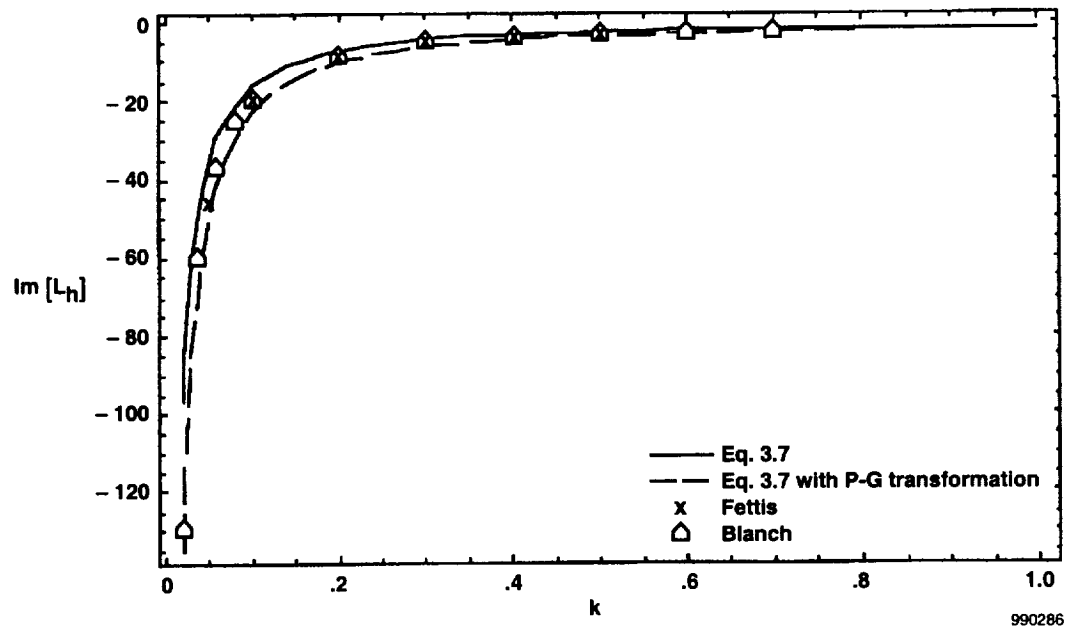


Figure 8. Lift coefficient due to plunge motion as a function of reduced frequency (k) with $M = 0.7$ (imaginary part).

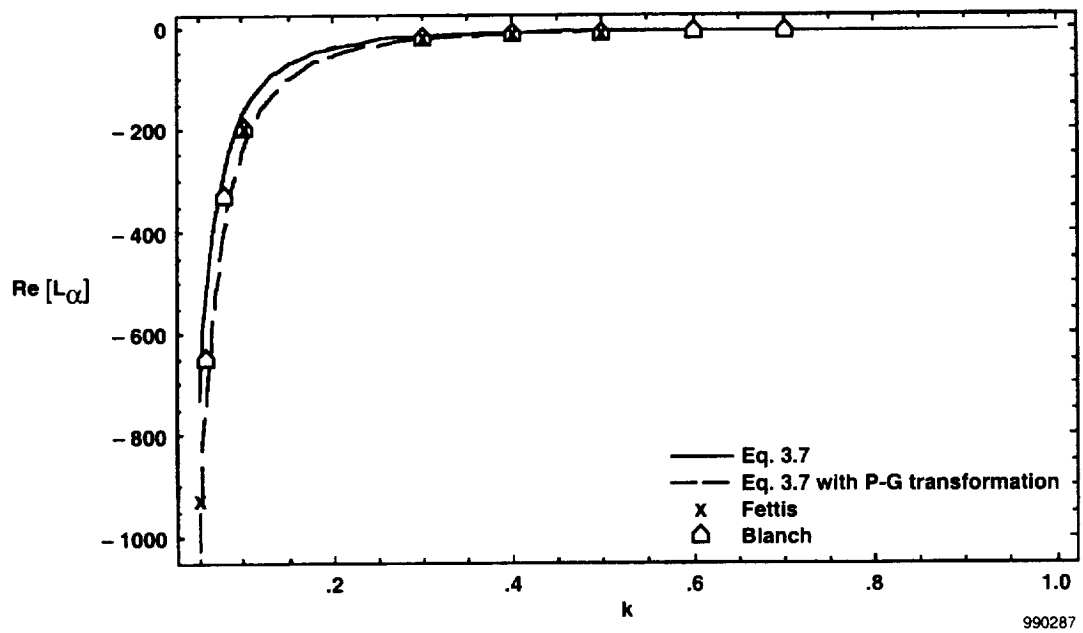


Figure 9. Lift coefficient due to pitch motion as a function of reduced frequency (k) with $M = 0.7$ (real part).

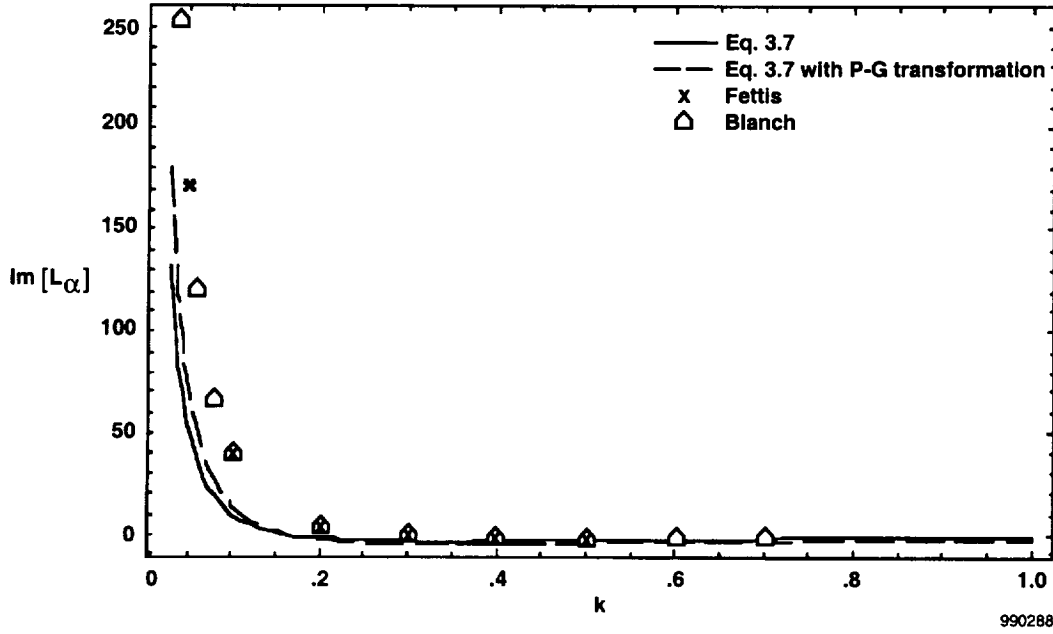


Figure 10. Lift coefficient due to pitch motion as a function of reduced frequency (k) with $M = 0.7$ (imaginary part).

5. APPLICATION TO FLUTTER ANALYSIS

A main motivation for calculating the lift and moment in unsteady aerodynamics is for use in aeroelastic stability analysis—flutter speed calculations. In this section, we calculate the “aeroelastic modes” (see reference 13, p. 550) of a wing with a large aspect ratio and demonstrate the simplicity and the ease of utility of the explicit lift and moment formulas found in section 3. We will also see the need for the Laplace transforms of lift and moment as opposed to the “oscillatory motion” values when calculating these “aeroelastic modes.”

We illustrate this with the continuum model of Goland¹⁷ for bending-torsion flutter for a uniform cantilever wing. The flutter speed has been calculated in references 17 and 18 for the incompressible case. Here we shall use our formulas to calculate the aerodynamic loading in the subsonic compressible case.

The dynamic equation for the wing is given as

$$\begin{aligned}
 m \frac{\partial^2 h(y,t)}{\partial t^2} + S_y \frac{\partial^2 \alpha(y,t)}{\partial t^2} + EI \frac{\partial^4 h(y,t)}{\partial y^4} &= -L(y,t), \\
 I_y \frac{\partial^2 \alpha(y,t)}{\partial t^2} + S_y \frac{\partial^2 h(y,t)}{\partial t^2} - GJ \frac{\partial^2 \alpha(y,t)}{\partial y^2} &= M(y,t), \\
 0 \leq t, \quad 0 \leq y \leq l,
 \end{aligned} \tag{5.1}$$

with boundary conditions of,

$$h(0,t) = \frac{\partial h(0,t)}{\partial y} = \alpha(0,t) = 0, \text{ and } \frac{\partial^2 h(l,t)}{\partial y^2} = \frac{\partial^3 h(l,t)}{\partial y^3} = \frac{\partial \alpha(l,t)}{\partial y} = 0.$$

Goland used the lift and moment of an infinitely spanned wing, but in actuality due to their dependence on h and α , the lift and moments in (5.1) do depend on y . For a wing with a sufficiently large aspect ratio, one can argue that the lift and moment can be written as the sum of the lift and moment of the infinitely spanned wing with a correction term that goes to zero as $y \rightarrow \infty$.¹⁹

Therefore, we proceed by taking the Laplace transform of equation (5.1) and denote the derivative with respect to y by superprime. We then have

$$\begin{aligned} m\lambda^2 \hat{h}(y,\lambda) + S_y \lambda^2 \hat{\alpha}(y,\lambda) + EI \hat{h}''''(y,\lambda) &= -\hat{L}(y,\lambda), \\ I_y \lambda^2 \hat{\alpha}(y,\lambda) + S_y \lambda^2 \hat{h}(y,\lambda) - GJ \hat{\alpha}''(y,\lambda) &= \hat{M}(y,\lambda), \\ 0 \leq y \leq l \end{aligned} \tag{5.2}$$

with boundary conditions:

$$\hat{h}(0,\lambda) = \hat{h}'(0,\lambda) = \hat{\alpha}(0,\lambda) = \hat{h}''(l,\lambda) = \hat{h}'''(l,\lambda) = \hat{\alpha}'(l,\lambda) = 0.$$

These equations yield the ‘‘aeroelastic’’ modes. The main point here is that we need $\hat{L}(y, \lambda)$, $\hat{M}(y, \lambda)$ given by (3.6), (λ is not purely imaginary). We may then proceed to calculate the real part of the aeroelastic mode corresponding to the first torsion mode, following reference 18, where the only difference is that the aerodynamics is restricted to the incompressible case, as in Goland. Figures 11 and 12 plot the real part of λ as a function of U , for the compressible as well as the incompressible case, at two different elevations. The flutter speed differs little from the incompressible case prior to flutter, even though the damping is higher initially. Tables 1 and 2 summarize the wing parameters and the flutter analysis. All of the reduced frequencies and Mach numbers at flutter condition are well within the range of validity of our approximation.

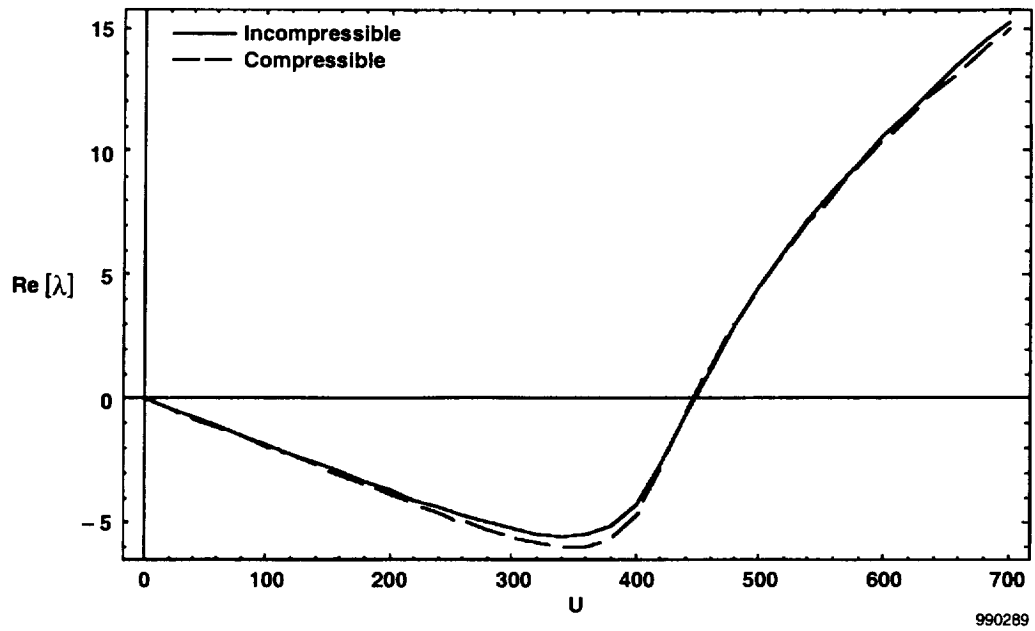


Figure 11. $\text{Re}[\lambda]$ as a function of air speed: Goland case at sea level.

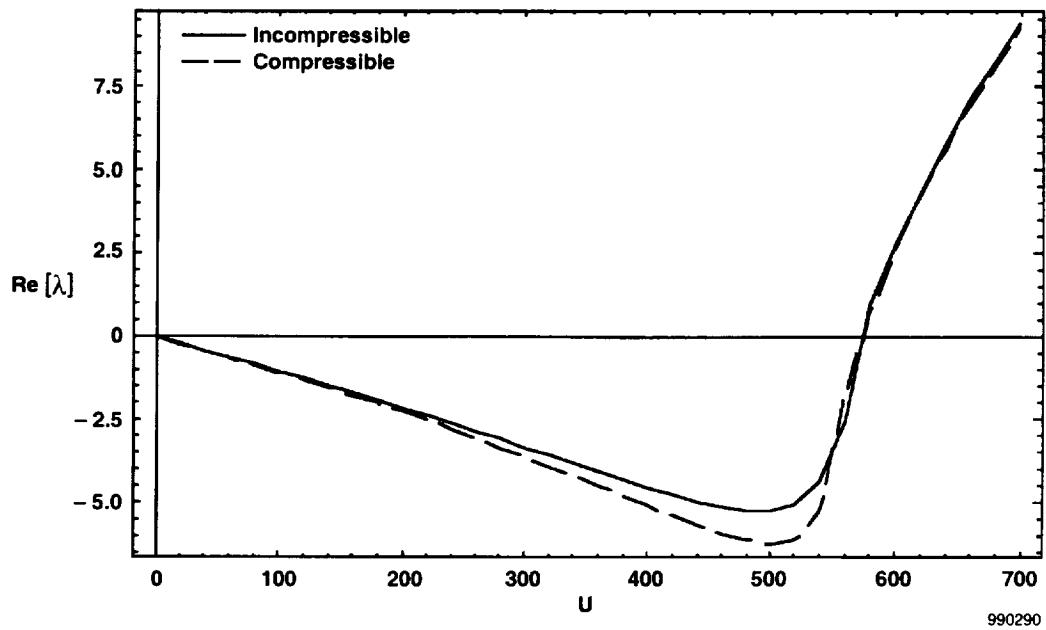


Figure 12. $\text{Re}[\lambda]$ as a function of air speed: Goland case at 20,000 ft.

Table 1. Parameters of the wing: Goland case.

Parameters	Values
m	0.746 slug/ft
S_y	0.447 slug/ft/ft
I_y	1.943 slug/ft ² /ft
EI	23.6×10^6 lb ft
GJ	2.39×10^6 lb ft
a	-1/3
l	20 ft
b	3 ft

Table 2. Flutter analysis of the wing: Goland case.

	Sea level			20 K ft above sea level		
	U_f (ft/sec)	k_f	M_f	U_f (ft/sec)	k_f	M_f
Incompressible	447	0.47	∞	574	0.36	∞
Compressible	446	0.46	0.40	574	0.35	0.55

6. CONCLUDING REMARKS

For application to aeroelastic analysis in the subsonic compressible regime, the current reliance on tabulated calculations of aerodynamic influence coefficients is cumbersome. In this paper, closed-form formulas of lift and moment coefficients are obtained based on the recent solution of the Possio equation by Balakrishnan.

For a Mach number less than 0.7 and reduced frequencies less than one, the closed-form formulas of lift and moment coefficients derived in this paper are within 10 percent of previous numerical results. As the Mach number gets closer to 0.7 and beyond, all methods of approximation fail. However, for such a high Mach number the aerodynamic flow is close to the transonic regime.

Numerical calculations are shown to be consistent with extant work. The ease of use of these formulas is demonstrated in the flutter analysis illustrated. No longer is there a need for “table look up” of aerodynamic coefficients in an iterative process of root finding.

REFERENCES

1. Possio, Camillo, “Aerodynamic Forces on an Oscillating Profile in a Compressible Fluid at Subsonic Speed.” *Aerotecnica*, vol. 18, 1938, pp. 441–458.
2. Dietze, F., “The Air Forces of the Harmonically Vibrating Wing at Subsonic Velocity (Plane Problem),” Parts I and II. U.S. Air Force translations, F-TS-506-RE and F-TS-948-RE, 1947. Originally appeared in *Luftfahrt-Forsch*, vol. 16, no. 2, 1939, pp. 84–96.
3. Fettis, Henry E., “Tables of Lift and Moment Coefficients for an Oscillating Wing-Aileron Combination in Two-Dimensional Subsonic Flow,” Air Force Technical Report 6688, 1951.
4. Haskind, M. D., “Oscillations of a Wing in a Subsonic Gas Flow,” Brown University translation, A9-T-22, 1948. Originally appeared in *Prikl. Mat. i Mekh.* vol. XI, no. 1, 1947, pp. 129–146.
5. Reissner, Eric, *On the Application of Mathieu Functions in the Theory of Subsonic Compressible Flow Past Oscillating Airfoils*, NACA TN-2363, 1951.
6. Timman, R., A. I. van de Vooren, and J. H. Greidanus, “Aerodynamics Coefficients of an Oscillating Airfoil in Two-Dimensional Subsonic Flow,” *Journal of the Aeronautical Sciences*, vol. 18, no. 12, December 1951, pp. 797–802.
7. Watkins, C. E., H. L. Runyan, and D. S. Woolston, *On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow*, NACA Report 1234, 1955.
8. Albano, E. and W. P. Rodden, “A Doublet-Lattice Method for Calculating Lift Distribution on Oscillating Surfaces in Subsonic Flows,” *AIAA Journal*, vol. 7, no. 2, February 1969, pp. 279–285.

9. Rodden, W. P., "The Development of the Doublet-Lattice Method," *Proceedings of the AIAA International Forum on Aeroelasticity and Structure Dynamics*, Rome, Italy, 1997.
10. Balakrishnan, A. V., *Unsteady Aerodynamics-Subsonic Compressible Inviscid Case*, NASA CR-1999-206583, August 1999.
11. Blanch, G., "Tables of Lift and Moment Coefficients for Oscillating Airfoils in Subsonic Compressible Flow," National Bureau of Standards Report 2260, 1953.
12. Küssner, H. G., "General Airfoil Theory," NACA TM-979, June 1941. Originally: "Allgemeine Tragflächentheorie." *Luftfahrt-Forsch*, vol. 17, no. 11-12, December 1940, pp. 370-378.
13. Bisplinghoff, Raymond L., Holt Ashley, and Robert L. Halfman, *Aeroelasticity*, Dover Edition, New York, 1996.
14. Küssner, H. G., and L. Schwarz, *The Oscillating Wing with Aerodynamically Balanced Elevator*, NACA TM-991, 1941. Translated from *Luftfahrt-Forsch*, vol. 17, no. 11-12, December 1940, pp. 337-354.
15. Smilg, Benjamin and Lee S. Wassermann, "Applications of Three-dimensional Flutter Theory to Aircraft Structures," Air Force Technical Report 4798, 1942.
16. Fetti, Henry E., "Reciprocal Relations in the Theory of Unsteady Flow Over Thin Airfoil Sections," *Proceedings of the Second Midwestern Conference on Fluid Dynamics*, Columbus, Ohio, 1952, pp. 145-154.
17. Goland, M., "The Flutter of a Uniform Cantilever Wing," *Journal of Applied Mechanics*, vol. 12, no. 4, December 1945, pp. A197-A208.
18. Balakrishnan, A. V., "Aeroelastic Control With Self-Straining Actuators: Continuum Models," *Proceedings of SPIE Smart Structures and Materials Conference 1998: Mathematics and Control in Smart Structures*, vol. 3323, San Diego, California, July 1998, pp. 44-54.
19. Reissner, E., *Effect of Finite Span on the Airfoil Distributions for Oscillating Wing-I, Theory*, NACA TN-1494, March 1947.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE April 2000		3. REPORT TYPE AND DATES COVERED Technical Memorandum
4. TITLE AND SUBTITLE Aerodynamic Lift and Moment Calculations Using a Closed-Form Solution of the Possio Equation			5. FUNDING NUMBERS WU 529 50 04 T2 RR 00 000	
6. AUTHOR(S) Jensen Lin and Kenneth W. Iliff				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) NASA Dryden Flight Research Center P.O. Box 273 Edwards, California 93523-0273			8. PERFORMING ORGANIZATION REPORT NUMBER H-2374	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) National Aeronautics and Space Administration Washington, DC 20546-0001			10. SPONSORING/MONITORING AGENCY REPORT NUMBER NASA/TM-2000-209019	
11. SUPPLEMENTARY NOTES Jensen Lin, University of California, Los Angeles, California and Kenneth W. Iliff, Dryden Flight Research Center, Edwards, California. Part of NASA DFRC grant NCC2-374 with UCLA.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT Unclassified—Unlimited Subject Category 02 This report is available at http://www.dfrc.nasa.gov/DTRS/			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words) In this paper, we present closed-form formulas for the lift and moment coefficients of a lifting surface in two-dimensional, unsteady, compressible, subsonic flow utilizing a newly developed explicit analytical solution of the Possio equation. Numerical calculations are consistent with previous numerical tables based on series expansions or ad hoc numerical schemes. More importantly, these formulas lend themselves readily to flutter analysis, compared with the tedious table-look-up schemes currently in use.				
14. SUBJECT TERMS Aeroelasticity, Lift and moment coefficients, Possio equation, Subsonic aerodynamics, Unsteady Aerodynamics			15. NUMBER OF PAGES 28	
			16. PRICE CODE A03	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT Unlimited	